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Rolling tachyon solution of two-dimensional string theory

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Abstract: We consider a classical (string) field theory of $c = 1$ matrix model which was developed earlier in [1] and subsequent papers. This is a noncommutative field theory where the noncommutativity parameter is the string coupling g_s. We construct a classical solution of this field theory and show that it describes the complete time history of the recently found rolling tachyon on an unstable D0 brane.

Keywords: Bosonic Strings, D-branes, Tachyon Condensation.

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1. Introduction

Recently there has been considerable interest in the c=1 matrix model arising from the identification of c=1 matrix model [2, 3] as a non-perturbative description of open string dynamics on unstable D0 branes of two-dimensional string theory (see [4, 5] for the fermionic version). The main merit of the matrix model description is that it provides a (holographic) non-perturbative formulation of two-dimensional string theory. The unstable D0 brane is identified with a non-relativistic fermion (the perturbative fluctuations correspond to relativistic fermions of the matrix model, see below).

The main point of Klebanov, Maldacena and Seiberg [3] is that it is necessary to treat the fermions quantum mechanically ($\hbar = g_s =$ finite), in order to obtain finite answers for quantities related to the decay of the D0 brane into closed strings. In this note we reinterpret the result of KMS in terms of the classical solutions of a field theory that is exactly equivalent to the c=1 matrix model. This field theory is 'noncommutative' because it takes into account the quantum mechanics of the fermions. Most of the formalism and the time dependent classical solution that we will discuss are known for some time [1, 9]. Here we will present an interpretation of the solution as a rolling tachyon. Our formalism

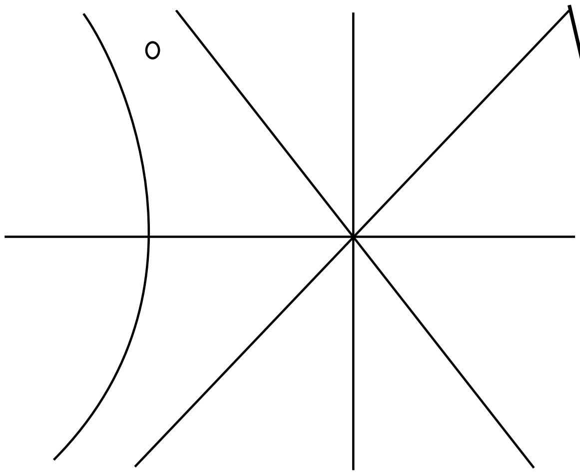

Figure 1: The classical solution u_1 represents a localized gaussian solution in phase space far from the Fermi level. At late times, it can be identified with ripples "near" the Fermi surface; the process can be interpreted as conversion into closed string modes.The support of the gaussian wave packet is nonzero because of uncertainty principle, with $\hbar \sim g_s$, giving a finite decay amplitude.

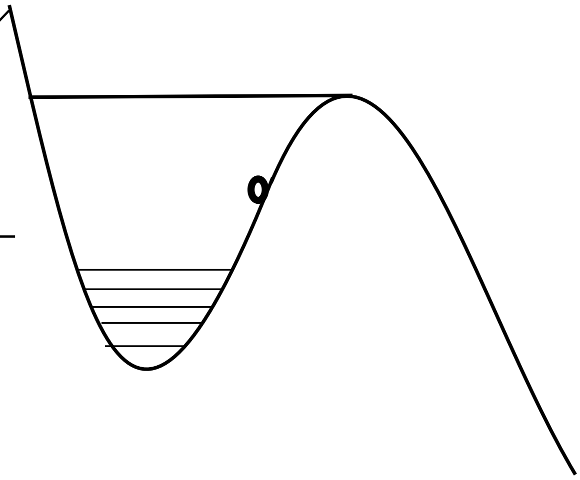

Figure 2: Same phenomenon in coordinate space. The finite localization in space implies finite width in energy.

enables us to write down the classical solution for all times. There appears a characteristic time scale $T_c = o(1)$ in the classical solution. For $t \ll T_c$ the solution can be identified with a D0 brane. For $t \gg T_c$ the solution can be identified as a perturbation of the filled Fermi sea which are directly mapped [8] to closed string tachyon fluctuations. It is worth emphasizing that the 'classical solutions' we are discussing incorporates a dependence on the string coupling g_s, because the field theory is non-commutative. They are different from the classical solutions of the underlying fermions, which are described by hyperbolas in a classical phase space (see section 2.). We include a review of relevant parts of our earlier work on matrix models in the appendix A–C.

In this paper we do not worry about the non-perturbative instability of string theory described by the potential given in figure 2. Our discussion can be easily adapted to the case of a symmetric potential (see [10], or for a fermionic interpretation, [4, 5]).

We would like to mention that in [12] there was an attempt to describe the rolling tachyon as a solution of a hybrid collective field theory, where the D0 brane collective coordinate is treated separately from the density waves near the Fermi level. In this treatment it was not possible to obtain the classical solution, representing the fermion density, for all times. It is not clear to us whether collective field theory [20] can, in principle, address this issue. This is because, in its present form, it does not seem to include the source terms of the open strings. For a different approach to noncommutativity in two dimensional string theory see [13]. While this paper was being written we received [14] which also discusses the rolling tachyon in the c=1 matrix model.

The matrix model picture of the unstable D0 brane has been used by [18] as an evidence for a new duality between open strings on unstable D branes and certain sectors of a closed string theory. In a sense our formulation of the classical two-dimensional string theory provides a description of both sides of the duality at two limits.

2. Classical analysis

We will first discuss the classical behaviour of the matrix model which is related to the $g_s \to 0$ limit of the discussion in the next section and also to the BCFT approach. In [3, 2] the following classical action for N D0 branes has been introduced, where $M_{ij}(t)$ describe the open string tachyon and $A_{0,ij}$ describes the gauge field:

$$S = \frac{1}{g_0} \int dt \;\, \mathrm{Tr} \left[\frac{1}{2}(D_t M)^2 + V(M) \right], \qquad D_t = \partial_t - i[A_0, \Big] \tag{2.1}$$

$$V(M) = -\frac{1}{2} M^2 + O(M^3) \tag{2.2}$$

$1/g_0$ is the D0-brane mass. The classical equations of motion are

$$\ddot{M} = M + O(M^2)\,, \qquad [M, \dot{M}] = 0\,.$$

The second equation is the Gauss law condition which reflects the gauge symmetry $M(t) \to U^\dagger(t) M(t) U(t), A_0(t) \to U^\dagger(t) A_0(t) U(t) + i U^\dagger(t) \partial_t U$. In the following we will fix the unitary gauge:

$$M(t) = \mathrm{diag}[q_i(t)]$$

If we ignore the $O(M^2)$ terms, the equation of motion becomes

$$\ddot{q}_i = q_i$$

with solution

$$q_i(t) = q_i \cosh t + v_i \sinh t \tag{2.3}$$

q_j, v_j can be interpreted as the initial position and velocity of the j-th D0 brane. Classically the D0 branes are non-interacting.

Hamiltonian: define the canonical momenta to be

$$p_i = \frac{v_i}{g_0}\,.$$

The hamiltonian becomes

$$H = \sum_i \frac{1}{2} \left(g_0 p_i^2 - \frac{q_i^2}{g_0} \right) = \frac{1}{g_0} \sum_i (\tilde{p}_i^2 - \tilde{q}_i^2)\,,$$

where $\tilde{p}_i = g_0 p_j, \tilde{q}_i = q_i$, so that $\{\tilde{p}_i, \tilde{q}_i\} = \delta_{ij} g_0$. Introducing the phase space density

$$u(\tilde{p}, \tilde{q}) = \sum_i \delta(\tilde{p} - \tilde{p}_i) \delta(\tilde{q} - \tilde{q}_i)$$

and the single particle hamiltonian

$$h(\tilde{p},\tilde{q}) = \frac{1}{2}(\tilde{p}^2 - \tilde{q}^2)$$

we can rewrite the hamiltonian as

$$H = \frac{1}{g_0}\int d\tilde{p}\; d\tilde{q}\; h(\tilde{p},\tilde{q})u(\tilde{p},\tilde{q})\,.$$

The range of $\tilde{p},\tilde{q}$, in the above expressions, is the entire plane. Hence each point in the $(\tilde{p},\tilde{q})$ plane is a possible classical state of the D0 brane of a given energy. We will see below that, in the quantum theory, the D0 branes are interacting. This has a drastic effect on the spectrum of allowed states.

If the $O(M^3)$ terms are included in the potential, the hyperbolic functions represent the initial behaviour. The q_i will have an oscillatory solution if it starts out inside the well on the left side (see figure 2), and will reach infinity if it is on the other side.

2.1 Rolling tachyon

The classical solution (2.3) is interpreted by [3, 2] as a rolling tachyon. In [3] the amplitude of such a configuration to emit closed strings is calculated (a) using BCFT (combining earlier rolling tachyon boundary state calculations of [15, 16] and Liouville theory boundary state calculations in [17]), and independently (b) using matrix model (where the asymptotically valid bosonization formulae for relativistic fermions were used to compare with BCFT). It is found that the expectation value of the total emitted energy diverges in the BCFT, whereas it appears to give a finite answer in the matrix model. In the next section we will discuss this in detail.

3. The fermion field theory or the $u(p,q,t)$ theory

The action (2.1) introduced above corresponds to the dynamics of the singlet sector of the $c = 1$ matrix model. As we reviewed in appendix A, the classical analysis presented above gets modified by interaction between the eigenvalues which come from the path integral measure, the result of which is that that the eigenvalues behave like fermions. As a result, the matrix model is described, in the double-scaling limit, by the second-quantized fermion action [6, 19] (see (A.9))

$$H = \frac{1}{g_s}\int dx\; \Psi^\dagger(x)\left[\frac{\hat{p}^2 - \hat{q}^2}{2} + 1\right]\Psi(x)\,, \qquad [\hat{q},\hat{p}] = ig_s \tag{3.1}$$

or by the bosonic variable $u(p,q,t)$ whose dynamics is given by the classical action (see (A.25))

$$\begin{aligned} S &= \int dt\, ds\, \frac{dpdq}{2\pi g_s} u(\partial_t u \star \partial_s u - \partial_s u \star \partial_t u) - \int dt \frac{dpdq}{2\pi g_s} u(p,q,t)(h(p,q)+1) \\ h(p,q) &= \frac{p^2-q^2}{2} \end{aligned} \tag{3.2}$$

and the constraints

$$\begin{aligned} u \star u &= u\,, \\ \int \frac{dpdq}{2\pi g_s} u(p,q) &= N\,. \end{aligned} \tag{3.3}$$

In the context of the double-scaled theory the last equation is interpreted appropriately in the limit $N \to \infty$ (see below, eq. (3.8),(3.13), (3.16)).

For our purposes here, the action will not play a role other than to yield the equation of motion

$$[\partial_t + (p\partial_q + q\partial_p)]\, u(p,q,t) = 0 \tag{3.4}$$

which follows from the variation of the action (3.2) (see appendix A).

The appearance of the star product (see (A.26)) indicates that the field theory of $u(p,q,t)$ is noncommutative, reflecting the noncommutative structure of the p,q plane. The noncommutativity parameter is the string coupling g_s. We will see below that it is essentially the noncommutative nature of this bosonic theory that prevents the divergence associated with the rolling tachyon.

Remarks on noncommutative solitons': the equation $u \star u = u$ has reappeared in the context of noncommutative solitons [21]. The projector solution (A.20) has also been rediscovered in that context. In the light of this development, the Fermi sea and the D-brane solution that we will describe below can be identified as rank N time independent and time dependent solitons of $c = 1$ theory (respectively).

3.1 The solution

We will now describe the solution of the above field theory that describes the rolling tachyon on the unstable D0-brane. First some preliminaries.

It is easy to solve the equation of motion (3.4)

$$u(p,q,t) = u_{\text{initial}}(\bar{p}(t), \bar{q}(t))\,, \tag{3.5}$$

where

$$(\bar{p}(t), \bar{q}(t)) = (p\cosh t - q\sinh t, -p\sinh t + q\cosh t)\,.$$

Note that time evolution preserves the area in phase space. Since the constraints are preserved by the equation of motion (easy to check) the simplest method of finding solutions to the equation of motion as well as the constraints is to construct $u_{\text{initial}}(p,q,0)$ satisfying the constraints and use (3.5).

We will construct various solutions by using the following observation [1, 9]: the constraints simply mean the rank N projector condition (see appendix A). Thus, we should first construct various rank N projection operators $\hat{u}$ in the single-particle Hilbert space and then convert it to $u(p,q)$ using (A.22).

We begin with the solution corresponding to the Fermi sea.

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Fermi sea:

$$\hat{u} = P_N = \sum_1^N |\chi_\nu\rangle\langle\chi_\nu| \equiv \hat{u}_0 \,.$$

It is straightforward to write down the corresponding function $u_0(p,q) = Tr(\hat{u}_0 \; w(p,q))$ (see (A.22)).

Small fluctuations around the Fermi level and closed strings: the small fluctuations or density waves around the Fermi level (energies small compared with g_s) are described by an effective boson theory that is described in appendix C (eq. (C.2)). This boson field ϕ is related to the closed string massless mode (tachyon) by the well known leg pole transform [22]–[24]. Since this sort of mapping is tied to the $p^\pm$ parameterization of the classical Fermi fluid profile which does not always work, in [8, 10] the closed string tachyon was mapped directly to a low energy fluctuation of the phase space density $\delta u(p,q,t) = u(p,q,t) - u_0(p,q)$:

$$\begin{aligned} T(x,t) = & \int \frac{dpdq}{2\pi g_s} G_1(x;p,q)\delta u(p,q,t) \\ & + \frac{1}{2}\int \frac{dpdq}{2\pi g_s}\int \frac{dp'dq'}{2\pi g_s} G_2(x;p,q;p',q')\; \delta u(p,q,t)\; \delta u(p',q',t) \\ & + \cdots . \end{aligned} \tag{3.6}$$

The precise forms of G_1, G_2 are given in [10]. As the fermion fluctuation moves far to the left (away from the turning point), the transform looks like

$$T(x,t) = \int \frac{dp\; dq}{2\pi g_s} f\left(-\frac{qe^{-x}}{\sqrt{g_s}}\right)\; \delta u(p,q,t) + O(xe^{-2x}) \tag{3.7}$$

where the function f is given by a Bessel function

$$f(\sigma) = \frac{1}{2\sqrt{\pi}} J_0\left(2\left(\frac{2}{\pi}\right)^{1/8}\sqrt{\sigma}\right).$$

Far away from the turning point, a matrix model fluctuation around q corresponds to a tachyon fluctuation roughly around $x \sim \ln(-q)$ but with a tail given by the above equation. The precise relation between matrix model fluctuations and tachyon fluctuations is both non-local and non-linear, as seen above.

The equation of motion of the tachyon and its interactions can be derived from the $u(p,q,t)$ dynamics [8, 10].It is in this process that we see the emergence of closed string backgrounds. In the present case the background is flat 2-dim. Minkowski spacetime and a linear dilaton. Departures from flat spacetime begin to appear (in the symmetric matrix model) depending upon how the Fermi sea is filled. In [10] an unequal filling of the Fermi sea on the two sides of the potential gave rise to a curved spacetime in 2-dims.which corresponds to the asymptotic form of the metric of the 2-dim. black hole [25]–[27].

We should make a comment about the constraint (3.3). Since both u_0 and u satisfy this constraint, (3.3) should be understood for the small fluctuations as

$$\int \frac{dp\; dq}{2\pi g_s} \delta u(p,q) = 0\,. \tag{3.8}$$

D-brane: we wish to describe a classical solution $u(p,q,t)$ which represents a localized fermion high above the Fermi sea (figure 2). Since the rank of $\hat{u}$ is always N (cf. (3.3)), we must lift a fermion from the Fermi sea and put it up. The first guess would be to put the fermion up in an energy eigenstate ψ_ν (of energy ν far above the Fermi sea)

$$\hat{u} = \hat{P}_{N-1} + |\psi_\nu\rangle\langle\psi_\nu| \,. \tag{3.9}$$

However, it is easy to see that energy *eigenstates* are not well-localized [28]. Thus for this fermion to be localized, it must have a wave-function $\psi(q,t)$ which is a linear combination of energy eigenstates. We will suppose that the wavefunction $|\psi(x,t)\rangle$ is such that the phase space location of the fermion is localized, within a size $\hbar = g_s$, around the point (q_0, p_0). Such a wavefunction is given by (at $t = 0$)

$$\psi(x,0) = \exp\left[-\frac{1}{2g_s}\left((x-q_0)^2 + 2ip_0x\right)\right].$$

The projector $|\psi\rangle\langle\psi|$ is not orthogonal to $\hat{P}_{N-1}$, however, since $|\psi\rangle$ is a linear combination of an infinite number of energy eigenstates, including those inside the Fermi sea! The naive solution (3.9) wouldn't satisfy $\hat{u}^2 = \hat{u}$, therefore.

The modification required is not difficult. We need to project out from $|\psi(x,0)\rangle$ the components along P_{N-1} as in Gram-Schmidt orthogonalization. The result is

$$u(p,q,t) = P_{N-1}(p,q) + u_1(p,q,t) - u_{01}(p,q,t)\,,$$

where

$$u_1 = \exp\left[-\frac{1}{2g_s}\left((\bar{q}(t)-q_0)^2 + (\bar{p}(t)-p_0)^2\right)\right] \tag{3.10}$$

and

$$\hat{u}_{01} = \hat{P}_{N-1}|\psi_1\rangle\langle\psi_1| + |\psi_1\rangle\langle\psi_1|\hat{P}_{N-1}\,.$$

We have used (A.22) to switch back and forth between operators and functions on phase space. It is easy to check that (App. B) u_{01} is negligible. We will therefore write our solution as

$$u(p,q,t) = u_0(p,q) + u_1(p,q,t)\,. \tag{3.11}$$

We have written $u_0(p,q)$ for $P_{N-1}(p,q)$ since in the double scaling limit they correspond to the same function (although the area is depleted by one, see (3.13)). u_1 represents a localized wave packet in phase space which is far from the D-brane at $t = 0$, representing a D-brane. The energy of u_1 is clearly

$$E_1 = \frac{1}{g_s}\left(\frac{p_0^2 - q_0^2}{2} + 1\right), \tag{3.12}$$

where the quantity in the parenthesis is of order one by choice.

It is clear from the above discussion that the solution $u_1(p,q,t)$ satisfies the constraint

$$\int \frac{dp\,dq}{2\pi g_s} u_1(p,q,t) = 1 \tag{3.13}$$

which is different from (3.8) satisfied by the small fluctuations (the reason is that the the background u_0 here is one-fermion depleted). For n D0 branes see subsection 3.4.

It is useful to look at the position space density corresponding to (3.11). We have

$$\rho(q,t) = \int \frac{dp}{2\pi g_s} u(p,q,t) = \rho_0 + \rho_1\,,$$

where $\rho(q) \to (1/g_s)\sqrt{q^2-1}$ as $g_s \to 0$ represents the fermion density in the sea, and

$$\rho_1(q,t) = \frac{1}{\sqrt{2\pi\Delta(t)}} \exp\left[-\frac{(q-\bar{q}_0(t))^2}{2\Delta(t)}\right],$$

where

$$\Delta(t) = \frac{g_s}{2}\cosh 2t,\ \bar{q}_0(t) = q_0\cosh t + p_0 \sinh t$$

represent, respectively, the time-dependent dispersion and the trajectory of the centre of the wave-packet.

The time T_c which needs to be crossed to reach the weak coupling region, i.e. $|\bar{q}_0(t)| \gg o(1)$, gives a characteristic time scale of the solution. Clearly

$$T_c = O(1)$$

(there is another time scale t_0 characterizing spreading of the wave packet $\Delta(t_0) \sim o(1)$; $t_0 = -\ln\sqrt{g_s}$).

- For $t \ll T_c$, the solution (3.11) represents the Fermi sea + 1 D0-brane.

  It is easy to see (appendix C) that at early times such as these the solution $u_1(p,q,t)$ does not satisfy the equations which describe the small fluctuations around the Fermi surface (effective coupling g_s/q must be small for such fluctuations). Alternatively, in this region the perturbative definition (3.6) or (3.7) breaks down.

- For $t \gg T_c$, by definition $|\bar{q}_0(t)| \gg 1$, hence the effective coupling is small. In terms of a phase space picture, note that

$$u_1(p,q,t) \sim \exp\left[-\frac{\left((p-q-p_0e^{-t})^2 + (p-q+q_0e^{-t})^2\right)}{(8g_se^{-2t})}\right].$$

  This describes a phase space density which is exponentially close to the asymptote $p=q$, and is also close to the Fermi level $p=\sqrt{q^2-\mu} \sim q$; this means that the phase space density can be represented as a small fluctuation of the Fermi surface. However, this argument ignores the delocalization of the $p+q$ variable. A more rigorous argument (see e.g. [6, 7]) is to note that non-relativistic corrections to energy levels ϵ above the Fermi surface are controlled by the norm of the wavefunction corresponding to the level ϵ. Therefore the corrections to the relativistic wavefunction go as $o(\epsilon/|q_0|^2)$ and become $\ll 1$ for $t \gg T_c$. Hence in this regime the fermion wave-packet can be decomposed into relativistic wave-functions.

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It is possible to represent $u_1(p,q,t)$ at such late times as closed string tachyon fluctuations using equations (3.6), (3.7). Equivalently, it can be shown that the solution we have found, at late times satisfies the equations of motion of density waves near the Fermi surface. The process of time-evolution of $u_1(p,q,t)$ can, therefore, be interpreted as decay of a D-brane into closed string modes.

3.2 Absence of divergence

Note that the solution that we have described here is *classical*, still the description of the decay is free of divergence, since (as [3] have already argued and is clear from above) a fermion with finitely localized wavefunction has a finite $\langle E_{\text{total}}\rangle \sim 1/g_s$. Such a description was not possible in the standard *classical* descriptions like BCFT, since the fermions had sharply localized position *and* momenta. Here, however, we have a classical (noncommutative) description, which is not limited in such a fashion. Indeed, it can describe a finite fuzz of the particle phase point because of the noncommutative nature of the classical field theory.

Our formulation here also shows how to understand the conversion of the D0 brane to tachyons by using (3.6). Using this equation (and its Fourier transform in the x-variable), we get the distribution of tachyon quanta at various energies at late times (it becomes a non-uniform distribution with finite total energy). The total energy indeed turns out to be $1/g_s$. The fact that at finite g_s the total energy of the tachyons is finite, and equal to (3.12), simply follows from the property that the hamiltonian of the tachyons is the same as the hamiltonian of the $u_1(p,q,t)$ variable (the transformation equations respect this fact). Since the energy of $u_1(p,q,t)$ is conserved, it is always given by (3.12), at early as well as late times. At late times the map to tachyons is available, hence the energy of the tachyon fluctuations can be equated to the energy of u_1, namely (3.12).

3.3 More general forms of u_1

It is not difficult to see that the specific gaussian form of the phase space density that we assumed above have not played any role for our purposes here. The main point is that any localized wave packet must satisfy the uncertainty relation $\Delta p\Delta q \sim g_s$. All our conclusions can be shown to follow from this property. However, the shape of the phase space density captures information about the quantum properties of the D0-brane beyond the classical property of position and velocity (q_0, p_0).

3.4 Multiple D0 branes

It is easy to generalize the above discussion to construct multiple D0 branes. Sticking to the gaussian form for simplicity, the solution is given by

$$\begin{aligned} u(p,q,t) &= u_0(p,q) + \delta u(p,q,t) \\ \delta u(p,q,t) &= \sum_{j=1}^{n} \exp\left[-\frac{1}{2g_s}\left((\bar{q}(t)-q_j)^2 + (\bar{p}(t)-p_j)^2\right)\right]. \end{aligned} \tag{3.14}$$

Here u_0 represents the Fermi sea depleted by n fermions from the top (in the double-scaled limit it coincides with the original Fermi sea). The centres of the gaussians are chosen such that each $p_i, q_i \sim o(1)$ and for each $i \neq j$

$$(p_i - p_j)^2 + (q_i - q_j)^2 \gg o(g_s)\,. \tag{3.15}$$

As before, it is trivial to see that (3.14) satisfies the equation of motion. The condition (3.3) in this case is

$$\int \frac{dp\; dq}{2\pi g_s} \delta u(p, q, t) = n \tag{3.16}$$

which is also easy to check.

The constraint (A.23) is more nontrivial. First, as in the case of (3.11), one needs to show that the overlap of each of these D0 branes with the Fermi sea is small; this follows in a manner similar to appendix B. In addition, one needs to show that the the overlap u_{ij} between each pair i, j of D-branes is small; it follows that

$$u_{ij} \sim \exp\left[-\frac{(p_i - p_j)^2 + (q_i - q_j)^2}{2g_s}\right].$$

This is small when the centres of the gaussians are chosen as in (3.15).

4. Conclusion

- We have constructed a solution of 2-dim string theory valid for arbitrary times. Before a characteristic time $T_c = o(1)$ it describes a D-brane (plus Fermi sea). Later, it describes ripples which can be translated into tachyon modes by appropriate integral transforms. This constitutes a classical description of the rolling tachyon which decays into closed string modes.

- The previous classical descriptions such as BCFT suffered from divergences because the phase space location was infinitely sharply localized in these classical descriptions. In the description presented here, the field theory is noncommutative (with noncommutativity parameter g_s). This allows for classical solutions with fuzzy initial conditions (fuzzy phase space locations), thereby leading to a finite result $\langle E_{total}\rangle = 1/g_s$. It is an essential feature of noncommutative field theory to incorporate g_s effects at the 'classical level'.

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A. Review of string field theory of c=1

We review salient features of the c=1 bosonic string field theory developed in [6, 1, 8, 9].

- Non-interacting fermions: The partition function of the $c = 1$ reads:

$$Z = \int DM \ \exp[-S] , \qquad S = \frac{1}{g_0} \int dt \, \mathrm{Tr} \left[\frac{1}{2} (\dot{M})^2 + V(M) \right] \Big),$$

$V(M) = -M^2/2 - M^3$. The difference from (2.1) is the absence of the gauge field A_0. The inclusion of the gauge field by [3] amounts to restricting to the singlet sector of the above partition function. In this sector the theory reduces to that of N eigenvalues of the matrix M. The result of integration over the angles is that the eigenvalues q_i behave as non-interacting fermions. Each fermion is subject to a single-particle hamiltonian

$$h = g_0 \frac{p^2}{2} + \frac{1}{g_0} \left(-\frac{q^2}{2} - q^3 \right) . \tag{A.1}$$

- Double-scaling: The single particle energy levels (eigenvalues of h), ignoring tunneling out of the well, are as shown in figure 2. The ground state of the matrix model is represented by the N-fermion state in which the first N levels are filled. The location of the Fermi level, called $-|\mu_N|$, depends on g_0. There is clearly a critical value g_c of g_0 at which $|\mu_N| \to 0$ i.e. the Fermi level reaches the maximum of the potential, signaling a singularity of the partition function $Z(g_0)$. The limit

$$g_0 - g_c \approx \frac{1}{2\pi} \mu_N \ln \mu_N \to 0 \,, \quad \text{or} \quad \mu_B \to 0 \,, \tag{A.2}$$

defines, therefore, a continuum limit of the random triangulation represented by the matrix model. Consider, on the other hand, the 't Hooft planar limit of the matrix model $N \to \infty, g_0 N = \bar{g}_0$ constant, in which only planar diagrams survive (genus zero). Double scaling is defined in which these two limits are taken together:

$$N \to \infty \,, \quad \mu_N \to 0 \text{ (or } g_0 \to g_c) \quad \text{such that} \quad \mu = N\mu_N = \text{constant} \,. \tag{A.3}$$

By setting up a WKB expansion of the wavefunctions of (A.1) (which gets arranged in powers of $1/\mu^2$), and identifying it with the genus expansion of string theory, it can be seen that

$$\mu = \frac{1}{g_s} \,, \tag{A.4}$$

where g_s is the string coupling.

In the double scaling limit, the fermions are described in terms by a scaled hamiltonian written using second-quantized fermions

$$H = \int dq \ \Psi^\dagger(q) [h(\hat{q}, \hat{p}) + |\mu|] \Psi(q) \,. \tag{A.5}$$

Here we have incorporated the information about the Fermi level using $|\mu|$ as a Lagrange multiplier. The single particle hamiltonian becomes quadratic (the cubic term scales away to zero)

$$h = \frac{1}{2} (\hat{p}^2 - \hat{q}^2) \,, \qquad [\hat{q}, \hat{p}] = i \,. \tag{A.6}$$

Rescaling: to understand the semiclassical limit (cf. (A.4)) it is appropriate to perform a rescaling

$$\hat{q} \to \sqrt{g_s}\hat{q}\,, \qquad \hat{p} \to \sqrt{g_s}\hat{p} \tag{A.7}$$

so that

$$[\hat{q},\hat{p}] = i g_s\,. \tag{A.8}$$

This indicates that in the limit of small g_s, the one-particle phase space can be thought of as cells of size g_s.

In this notation, the hamiltonian becomes

$$H = \frac{1}{g_s}\int dx\ \Psi^\dagger(x)[h+1]\Psi(x) \tag{A.9}$$

with h still given by (A.6).

We will denote energy levels of h by

$$h\chi_\nu(x) = \nu\chi_\nu(x)\,, \tag{A.10}$$

where $\chi_\nu(x)$ span the single particle Hilbert space $\mathcal{H}_1$. The Fermi level is defined by the wavefunction $\chi_\nu(x)$ with energy

$$\nu = -1\,. \tag{A.11}$$

A.1 Construction of the bosonic field theory

Example of finite number of single-particle levels: let us consider N non-interacting fermions each of which can occupy K levels. Note that (A.10) have infinite number of levels, but for simplicity we will consider first the case of K, N finite. The limit $K, N \to \infty$ will be taken afterwards. The states of a K-level system with N levels filled can be described in terms of the following overcomplete basis (coherent states):

$$|\Psi_g\rangle = \hat{g}|F_N\rangle\,. \tag{A.12}$$

Here $|F_N\rangle$ is the filled Fermi sea, g is a $\mathrm{U}(K)$ group element $\exp[i\theta_{\mu\nu}T_{\mu\nu}]$ and $\hat{g}$ is its representation in the many-fermion system $\exp[i\theta_{\mu\nu}J_{\mu\nu}]$. Here

$$\hat{J}_{\mu_\nu} = c^\dagger_\mu c_\nu \tag{A.13}$$

form a representation of $\mathrm{U}(K)$ in the fermion Fock space. $c_\nu, c^\dagger_\nu$ annihilates or creates (resp.) the a fermion in the single-particle state $\chi_\nu(x)$.

It is clear that a path integral over the many-fermion system can be converted to an integral over the group elements g, where g varies over the coset $\mathrm{U}(K)/H$ where the subgroup $H = \mathrm{U}(N)\times\mathrm{U}(K-N)$ reflects the invariance group of the filled Fermi sea in (A.12). Thus a (bosonic) description of the classical configuration space can be provided by such group variables (see spin-half example below). However, an

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alternative bosonic description is provided by the variables

$$u_{\mu\nu} = \langle \Psi | \hat{g} J_{\mu\nu} \hat{g}^\dagger | \Psi \rangle = (g \bar{u} g^\dagger)_{\mu\nu} \tag{A.14}$$

with

$$\bar{u}_{\mu\nu} = \langle F_N | J_{\mu\nu} | F_N \rangle \,. \tag{A.15}$$

To elucidate, let us briefly consider the example of a spin-half particle.

Case of spin half particle: let us consider the example of $K = 2$, $N = 1$, i.e. a two-level system with half-filling. Two orthogonal basis states are $|F_1\rangle \equiv c_1^\dagger |0\rangle$ and $c_2^\dagger |0\rangle = c_2^\dagger c_1 |F_1\rangle$. Here $c_1^\dagger, (c_2^\dagger)$ create a particle in the state 1 (2 resp.); $|0\rangle$ is the no-particle state. These form a spin-1/2, charge 1, representation of $\mathrm{U}(2) = \mathrm{SU}(2) \times \mathrm{U}(1)$ under the representation (A.13). The coherent state (A.12) in this case can be identified with the unit vector $\mathbf{n}$ in R^3 obtained by applying the rotation g on a given unit vector, say $\bar{\mathbf{n}} = (0, 0, 1)$. The set of vectors $\mathbf{n}$ parameterize $\mathrm{U}(2)/(\mathrm{U}(1) \times \mathrm{U}(1)) = S^2$, the classical configuration space of a spinning particle. The quantity $u_{\mu\nu}$ in (A.14) is simply related to this spin variable:

$$u_{\mu\nu} = \mathbf{n}.\vec{\sigma}_{\mu\nu} + \delta_{\mu\nu} \,.$$

To see this, use the formula:

$$\langle \mathbf{n} | \mathbf{J} | \mathbf{n} \rangle = -j \mathbf{n} \,,$$

where in our case $j = 1/2$. The base value $\bar{u}$ in(A.14) corresponds to the representative value of $\mathbf{n}$ in the orbit. Here $\vec{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\}$ are the Pauli matrices.

A spin-half particle moving in a magnetic field $\mathbf{B}$ translates to the following fermion equation:

$$\partial_t c^\mu(t) = \mathbf{B}.\vec{\sigma}_{\mu\nu} c^\nu(t) \,.$$

The bosonic EOM is more familiar:

$$\partial_t \mathbf{n} = \mathbf{n} \times \mathbf{B}$$

obtainable from an action

$$S = \int dt\ ds\, \mathrm{Tr}[u(\partial_t u \partial_s u - \partial_s u \partial_t u) + Bu] \,, \tag{A.16}$$

where $B = \mathbf{B}.\sigma$. Here $u(t, s)$ is a one-dimensional extension of the classical variable $u(t)$. It is easy to visualize in terms of the equivalent variable $\mathbf{n}(t)$; if $\mathbf{n}(t)$ traces a closed path in the configuration space $S2$, then $\mathbf{n}(t, s)$ describes the solid angle enclosed by the closed path. An infinitesimal variation of $\mathbf{n}(t, s)$ (which is an infinitisimal $SU(2)$ transformation) involves only the boundary curve $\mathbf{n}(t)$, giving rise to the equation of motion for the original variable $\mathbf{n}(t)$ written above.

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Let us come back to the case of general K, N. We now understand the u-variables as a classical spin variable, characterizing a G-(coadjoint)-orbit, ($G = U(K)$), of the representative value (A.15). Using (A.13), the latter evaluates to

$$\bar{u}_{\mu\nu} = \sum_{\lambda=1}^{N} \delta_{\lambda\mu}\delta_{\lambda\nu} = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \cdot & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & & & & & \cdot & \\ & & & & & & 0 \end{pmatrix}, \tag{A.17}$$

where the first N diagonal entries are 1 and rest are 0. It is useful to regard the matrix $u_{\mu\nu}$ as an operator in the first quantized Hilbert space $\mathcal{H}_1$: thus

$$u_{\mu\nu} =: \langle\mu|u|\nu\rangle\,, \qquad \bar{u}_{\mu\nu} =: \langle\mu|\bar{u}|\nu\rangle\,. \tag{A.18}$$

Equation (A.17) then implies that $\bar{u}$ is a rank N projector (onto the first N single-particle levels)

$$\bar{u} = P_N = \sum_{\lambda=1}^{N} |\lambda\rangle\langle\lambda|\,. \tag{A.19}$$

This implies

$$u = gP_N g^\dagger = \sum_{\lambda=1}^{N} |\lambda_g\rangle\langle\lambda_g|\,, \qquad |\lambda_g\rangle \equiv g|\lambda\rangle\,, \tag{A.20}$$

where the $U(K)$ matrix g is interpreted to act on $\mathcal{H}_1$ in a manner similar to that in (A.18). The common property of the orbit (A.20) is that all u's are rank N projectors, which is equivalent to the equations

$$u^2 = u, \quad \mathrm{Tr}\, u = N\,. \tag{A.21}$$

Limit $K \to \infty$ or $c = 1$ model. We now return to $c = 1$. Here the single-particle Hilbert space $\mathcal{H}_1$ is infinite dimensional $K \to \infty$. The limiting case of $U(K)$ is identified with the group of unitary operators $U(\mathcal{H}_1)$, and is called the group $W(\infty)$. The rest of the analysis is pretty much unchanged.

Operators $\hat{u}$ on the one-particle Hilbert space $\mathcal{H}_1$ have an additional "phase space" representation (the Moyal map)

$$\begin{aligned} u(p,q) &= \mathrm{Tr}(\hat{u}\hat{w}(p,q)) \\ \hat{w}(p,q) &= \exp\left[\frac{i}{g_s}(q\hat{p} + p\hat{q})\right]. \end{aligned} \tag{A.22}$$

The operators $\hat{w}(p,q)$ provide a basis of $W(\infty)$ (see [1] where we have used the notation $\hat{g}(p,q)$ for $\hat{w}(p,q)$).

In terms of u(p,q), and switching to the rescaled phase space coordinates (A.7) the constraints (3.3) become

$$u \star u = u \tag{A.23}$$

$$\int \frac{dpdq}{2\pi g_s} u(p,q) = N \,. \tag{A.24}$$

The dynamics of the fermion system can be written entirely in terms of the u-variables [1](cf. (A.16)):

$$S = \int dt\ ds\ \frac{dpdq}{2\pi g_s} \left(u(\partial_t u \star \partial_s u - \partial_s u \star \partial_t u) + (h(p,q)+1)u(p,q) \right) , \tag{A.25}$$

where u satisfies the two constraint equations above. Here $u(p,q,t,s)$ is an extension of the variable as described below (A.16). Infinitesimal variation of $u(p,q,t,s)$ involves only the "boundary curve" $u(p,q,t,s)$ (see discussion below (A.16)), leading to the equation of motion (3.4). (Here the variation of $u(p,q,t)$ is an infinitisimal $W(\infty)$ transformation, $\delta u = \epsilon \star u - u \star \epsilon$. This variation also leaves the constraints (3.3) invariant.)

This constitutes a *noncommutative bosonic field theory* which describes the c=1 matrix model. The star product, explicitly, turns out to be

$$(A \star B)(p,q) = \exp\left[i\frac{g_s}{2}(\partial_q \partial_{p'} - \partial_p \partial_{q'}) \right] (A(p,q)B(p',q'))|_{p=p',q=q'} \,. \tag{A.26}$$

Role of fermions: the constraint (A.23) simply reflects the projection operator structure (A.17), which in turn reflects the fermionic nature of the problem. The only non-zero diagonal entries are 1 because of Pauli exclusion principle.

B. Computation of u_{01}

The wavefunctions $\chi_{-\nu}(x)$ appearing in (A.10) have the following asymptotic form [11] relevant for our purpose:

$$\nu \gg x^2 : \quad \psi_\nu(x) \sim \nu^{-1/4} \exp[ix\sqrt{\nu}] \,. \tag{B.1}$$

To get the Fermi level we put the condition (A.11). Our wavefunction ψ_1 is given by

$$\psi_1(x) = \exp\left[-\frac{(x-q_0)^2}{2} + ip_0 x \right] .$$

It is trivial to calculate the overlap:

$$\langle \psi_1 | \psi_\nu \rangle \propto \nu^{-1/4} \exp\left[-\frac{(\sqrt{\nu}+p_0)^2}{2} \right] \exp[iq_0\sqrt{\nu}] \,.$$

For $\nu = \mu \sim 1/g$, and $p_0, q_0 \sim O(1)$ this goes as

$$\exp\left[-\frac{1}{g_s} \right] .$$

C. u_1 is not describable by collective field at early times

The collective field limit:

In the limit $g_s \to 0$ the equation $u \star u = u$ becomes $u^2 = u$, which means that $u(p,q,t)$ is 1 (in the region occupied by the fermions) or else zero. Such $u(p,q,t)$ are in simple cases described by two p-extremities $p_+(q,t), p_-(q,t)$ of the region occupied by the fermions [22]. In this case, the various moments of the phase space density:

$$\begin{aligned} m_0(q,t) &= \int \frac{dp}{2\pi g} u(p,q,t) = p_+(q,t) - p_-(q,t) \\ m_1(q,t) &= \int \frac{dp}{2\pi g}\ p\ u(p,q,t) = \frac{1}{2}(p_+^2(q,t) - p_-^2(q,t)) \\ m_2(q,t) &= \int \frac{dp}{2\pi g}\ p^2\ u(p,q,t) = \frac{1}{3}(p_+^3(q,t) - p_-^3(q,t)) \\ \cdots &= \cdots \end{aligned} \tag{C.1}$$

all get related to the two functions $p^{\pm}(q,t)$ and hence they get related to the first two, which are

$$m_0(q,t) = \rho(q,t)\,, \qquad m_1(q,t) = \Pi(q,t)\rho(q,t)\,.$$

This is the situation for $g_s \to 0$ and for fluctuations not far from the Fermi sea. In this case, the dynamics of such fluctuations is given by the following effective action (the collective coordinate action) [20, 6, 19, 22]:

$$S = \int dt\ dx \left[\partial_+\phi\partial_-\phi - \frac{\pi}{6g_s}\frac{1}{\sinh 2x}\{(\partial_+\phi)^3 - (\partial_-\phi)^3\}\right], \tag{C.2}$$

where $\partial_\pm\phi$ are related to ρ, Π.

It is easy to see that moments of $u_1(p,q,t)$ do not satisfy the equation of motion that follow from (C.2) for $t \ll g_s$ [9]. The extra terms leading to the disagreement persist at large t. They are small but nonzero.

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